Math 266.3 Midterm Solutions

1. 20% Determine an equation of the quadratic polynomial which passes through the points

$(-1,1)$, $(1,-5)$, and $(2,-2)$.

Solution: $y = ax^2 + bx + c$ must be satisfied at the three points, so we must have:

$1 = a(-1)^1 + b(-1) + c$

$-5 = a(1)^2 + b(1) + c$

$-2 = a(2)^2 + b(2) + c$ or

$a - b + c = 1$
$a + b + c = -5$
$4a + 2b + c = -2$

Using an augmented matrix we have

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -5 \\ 4 & 2 & 1 & -1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & -6 \\ 0 & 6 & -3 & -5 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -3 & 13 \end{array}\right] \to$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & \frac{16}{3} \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -\frac{13}{3} \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & \frac{7}{3} \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -\frac{13}{3} \end{array}\right]$$

so the required quadratic polynomial is $y = \frac{7}{3}x^2 - 3x - \frac{13}{3}$

2. 10% Compute the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 5 & 8 \end{bmatrix}$.

Solution: $A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 5 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & -1 & 5 & -8 \\ 0 & 1 & 5 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has rank 2.

3. 20% If $\mathsf{B} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, then find the determinant of B, $\det(\mathsf{B})$, by using the cofactor expansion in the first column.

Solution: $\det(\mathsf{B}) = 1 \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} =$

$[(3)(2) - (-1)(-1)] + [(1)(2) - (-1)(1)] + [(1)(-1) - (3)(1)] = 5 + 3 + (-4) = 4$

4. If $\mathsf{C} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, then

(a) 10% find C^2

Solution: $\mathsf{C}^2 = \begin{bmatrix} 0 & 4 \\ -4 & 8 \end{bmatrix}$

(b) 20% Find C^{-1} by using an augmented matrix, and indicate which row operations are being performed at each step.

Solution: $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \to$ adding first row to second:

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 4 & 1 & 1 \end{array}\right] \to$ multiplying second row by :$\frac{1}{4}$

$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} \end{array}\right] \to$ subtracting second row from first row :

$\left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{4} & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} \end{array}\right]$, so $\mathsf{C}^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

(c) 10% Determine whether or not $\mathsf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathsf{v} = \begin{bmatrix} 2 \\ -11 \end{bmatrix}$ are eigenvectors of C. Justify your answers.

Solution: $\mathsf{Cu} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\mathsf{u}$, so u is an eigenvector.

$\mathsf{C}\mathsf{v} = \begin{bmatrix} -9 \\ -35 \end{bmatrix} = -4.5\mathsf{v} + \begin{bmatrix} 0 \\ -84.5 \end{bmatrix}$, so v is not an eigenvector.

5. 10% Find all solutions of the system of equations $\begin{array}{rcrcrcr} x & + & 2y & - & 4z & = & 10 \\ 2x & - & y & + & 2z & = & 5 \\ x & + & y & - & 2z & = & 7 \end{array}$

Solution: Using an augmented matrix, we have:

$$\left[\begin{array}{rrr|r} 1 & 2 & -4 & 10 \\ 2 & -1 & 2 & 5 \\ 1 & 1 & -2 & 7 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -4 & 10 \\ 0 & -5 & 10 & -15 \\ 0 & -1 & 2 & -3 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -4 & 10 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 6 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ so we have $x = 6$, $y = 3 + 2z$.